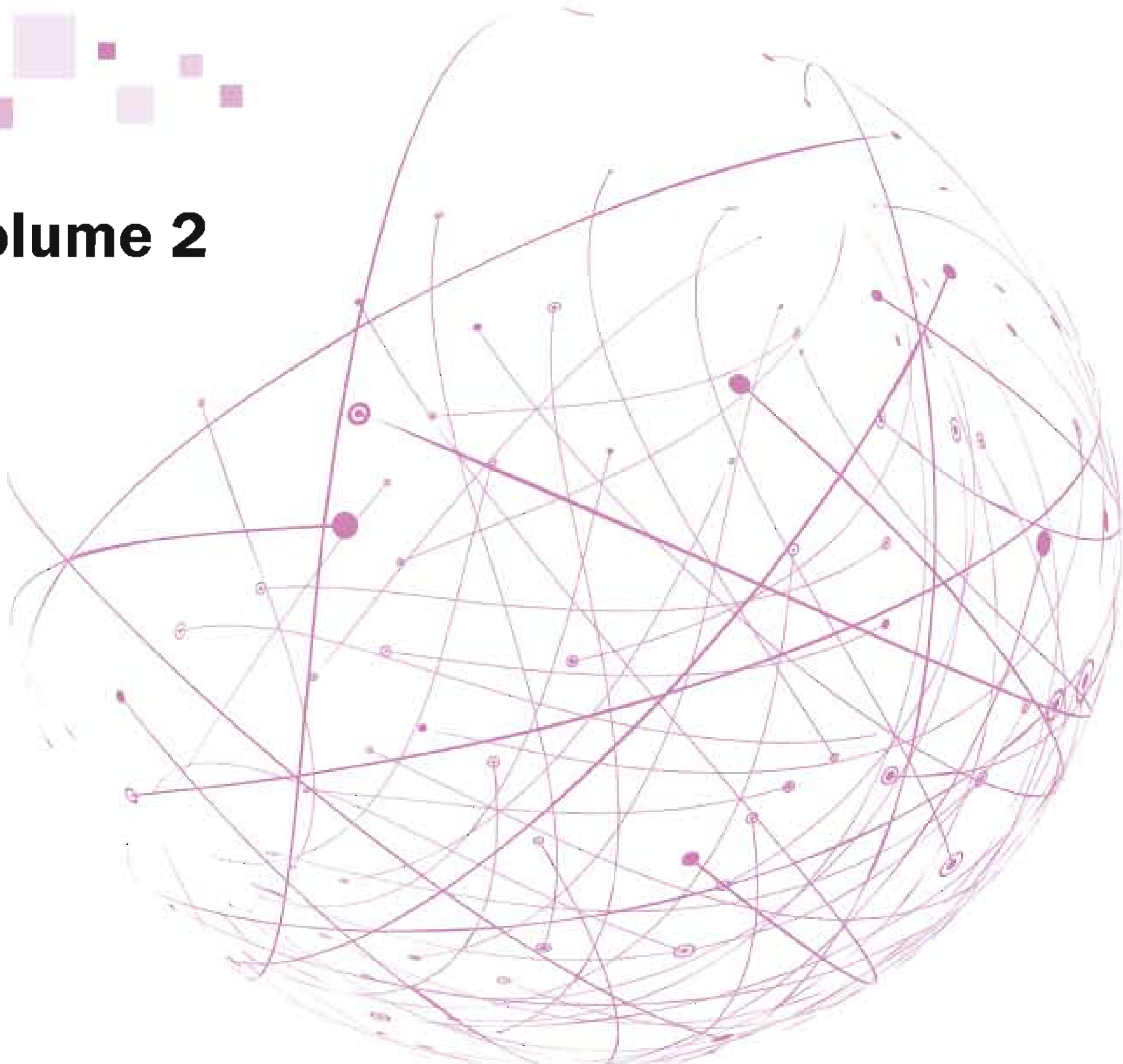
Sergey Y. Yurish
Editor

# Advances in Networks, Security and Communications: Reviews



## Volume 2

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# Chapter 5
# Mathematical Aspects of Neural Networks: Stability of Equilibrium Points

**Zlatinka Covacheva and Valéry Covachev**

## 5.1. Introduction

Anyone can see that the human brain is superior to a digital computer at many tasks. For example, from the processing of visual information point of view; a one-year-old baby is much better and faster at recognizing objects, faces, and so on than even the most advanced fastest supercomputer systems. The following reasons are the real motivation for studying neural computation [1]. It is an alternative computational paradigm to the usual one (based on a programmed instruction sequence), which was introduced by von Neumann [2] and has been used as the basis of almost all machine computation to date. The brain has many other features that would be desirable in artificial systems:

- It is robust and fault tolerant. Nerve cells in the brain die every day without affecting its performance significantly.
- It is flexible. It can easily adjust to a new environment by "learning", no need to be programmed in Pascal, Fortran or C+ and so on.
- It can deal with information that is fuzzy, probabilistic, noisy, or inconsistent.
- It is highly parallel. It is small, compact, and dissipates very little power.

Artificial neural networks are computational paradigms which implement simplified models of their biological counterparts, biological neural networks. Biological neural networks are the local assemblages of neurons and their dendritic connections that form the (human) brain. Accordingly, artificial neural networks are characterized by

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- Local processing in artificial neurons (or processing elements);
- Massively parallel processing, implemented by rich connection pattern between processing elements;
- The ability to acquire knowledge via learning from experience;
- Knowledge storage in distributed memory, the synaptic processing element connections.

Neural network simulations appear to be a recent development. However, this field was established before the advent of computers, and has survived at least one major setback and several eras. Many important advances have been boosted by the use of inexpensive computer emulations. Following an initial period of enthusiasm, the field survived a period of frustration and disrepute. Currently, the neural network field enjoys a resurgence of interest and a corresponding increase in funding [3].

The first artificial neuron was produced in 1943 by the neurophysiologist Warren McCulloch and the logician Walter Pitts [4]. But the technology available at that time did not allow them to do too much. Neural networks process information in a similar way the human brain does. The network is composed of a large number of highly interconnected processing elements (neurons) working in parallel to solve a specific problem. Neural networks learn by example. Much is still unknown about how the brain trains itself to process information, so theories abound.

An artificial neuron is a device with many inputs and one output. The neuron has two modes of operation, the training mode and the using mode. In the training mode, the neuron can be trained to fire (or not), for particular input patterns. In the using mode, when a taught input pattern is detected at the input, its associated output becomes the current output. If the input pattern does not belong in the taught list of input patterns, the firing rule is used to determine whether to fire or not.

An important application of neural networks is pattern recognition. Pattern recognition can be implemented by using a feed-forward neural network that has been trained accordingly. During training, the network is trained to associate outputs with input patterns. When the network is used, it identifies the input pattern and tries to output the associated output pattern. The power of neural networks comes to life when a pattern that has no output associated with it, is given as an input. In this case, the network gives the output that corresponds to a taught input pattern that is least different from the given pattern.

The above neuron does not do anything that conventional computers do not already do. A more sophisticated neuron is the McCulloch and Pitts model (MCP). The difference from the previous model is that the inputs are "weighted", the effect that each input has at decision making is dependent on the weight of the particular input. The weight of an input is a number which when multiplied with the input gives the weighted input. These weighted inputs are then added together and if they exceed a pre-set threshold value, the neuron fires. In any other case the neuron does not fire. In mathematical terms, the neuron fires if and only if

$$\sum_{i=1}^{m} X_i W_i > T,$$

where $W_i, i = \overline{1,m}$, are weights, $X_i, i = \overline{1,m}$, inputs, and $T$ is a threshold. The addition of input weights and of the threshold makes this neuron a very flexible and powerful one. The MCP neuron has the ability to adapt to a particular situation by changing its weights and/or threshold. Various algorithms exist that cause the neuron to "adapt"; the most used ones are the Delta rule and the back error propagation. The former is used in feed-forward networks and the latter in feedback networks.

The attempt of implementing neural networks for brain-like computations like pattern recognition, decision making, motor control and many others is made possible by the advent of large scale computers in the late 1950's. Indeed, artificial neural networks (ANN) can be viewed as a major new approach to computational methodology since the introduction of digital computers.

Although the initial intent of ANN was to explore and reproduce human information processing tasks such as speech, vision, and knowledge processing, ANN also demonstrated their superior capability for classification and function approximation problems. This has great potential for solving complex problems such as systems control, data compression, optimization problems, pattern recognition, and system identification.

Neural networks have wide applicability to real world business problems. In fact, they have already been successfully applied in many industries. Since neural networks are best at identifying patterns or trends in data, they are well suited for prediction or forecasting needs including: sales forecasting, industrial process control, customer research, data validation, risk management, target marketing.

ANN are also used in the following specific paradigms: recognition of speakers in communications; diagnosis of hepatitis; recovery of telecommunications from faulty software; interpretation of multi-meaning Chinese words; undersea mine detection; texture analysis; three-dimensional object recognition; hand-written word recognition; and facial recognition.

## 5.2. Hopfield Neural Networks

Hopfield-type (additive) networks have been studied intensively for more than two decades and have been applied to optimization problems [5-8] and [9]. The original model used two-state threshold "neurons" that followed a stochastic algorithm: each model neuron $i$ had two states, characterized by the values $V_i^0$ or $V_i^1$ (which may often be taken as 0 and 1, respectively). The input of each neuron came from two sources, external inputs $I_i$ and inputs from other neurons. The total input to neuron $i$ is then

$$\text{Input to } i = H_i = \sum_{i \neq j} T_{ij} V_j + I_i,$$

where $T_{ij}$ can be biologically viewed as a description of the synaptic interconnection strength from neuron $j$ to neuron $i$. The motion of the state of a system of $N$ neurons in state space describes the computation that the set of neurons is performing. A model therefore must describe how the state evolves in time, and the original model describes this in terms of a stochastic evolution. Each neuron samples its input at random times. It changes the value of its output or leaves it fixed according to a threshold rule with thresholds $U_i$ [10, 11]:

$$V_i \to V_i^0 \quad \text{if} \quad \sum_{i \neq j} T_{ij} V_j + I_i < U_i,$$
$$V_i \to V_i^1 \quad \text{if} \quad \sum_{i \neq j} T_{ij} V_j + I_i > U_i$$

In order to solve problems in the fields of optimization, neural control and signal processing, neural networks have to be designed such that there is only one equilibrium point and this equilibrium point is globally asymptotically stable so as to avoid the risk of having spurious equilibria and local minima. In the case of global stability, there is no need to be specific about the initial conditions for the neural circuits since all trajectories starting from anywhere settle down at the same unique equilibrium. If the equilibrium is exponentially asymptotically stable, the convergence is fast for real-time computations. The unique equilibrium depends on the external stimulus. The nonlinear neural activation functions $f_i, i = \overline{1, m}$, are usually chosen to be continuous and differentiable nonlinear sigmoid functions satisfying the following conditions:

a) $f_i( x) \to \mp\infty$ as $x \to \mp\infty$;

b) $f_i(x)$ is bounded above by 1 snd below by $-1$;

c) $f_i(x) = 0$ at a unique point $x = 0$;

d) $f_i'(x) > 0$ and $f_i'(x) \to 0$ as $x \to \mp\infty$;

e) $f_i'(x)$ has a global maximum value of 1 at a unique point $x = 0$.

Some examples of activation functions $f_i(\cdot)$ are

$$f_i(x) = \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad f_i(x) = \frac{1 - e^{-x}}{1 + e^{-x}} = \tanh(x/2),$$
$$f_i(x) = \frac{2}{\pi} \tan^{-1}\left(\frac{\pi}{2} x\right), \qquad f_i(x) = \frac{x^2}{1 + x^2} \operatorname{sgn}(x),$$

where $\operatorname{sgn}(\cdot)$ is a signum function and all the above nonlinear functions are bounded, monotonic and nondecreasing functions. It has been shown that the absolute capacity of an associative memory network can be improved by replacing the usual sigmoid activation functions. There, it seems appropriate that nonmonotonic functions might be better candidates for neuron activation in designing and implementing an artificial neural network. In many electronic circuits, amplifiers that have neither monotonically increasing nor continuously differentiable input-output functions are frequently adapted.

In [12] the global stability characteristic of a system of equations modelling the dynamics of additive Hopfield-type neural networks both in the continuous- and discrete-time cases is investigated. In particular, a novel method of obtaining a discrete-time dynamical system whose dynamics is inherited from the continuous-time dynamical system is studied. This aspect is important since numerical algorithms of Hopfield-type differential equations lead to discrete-time dynamic systems and such discrete-time systems should not give rise to any spurious behaviour if either system is to be used for coding equilibrium as associative memories corresponding to temporally uniform external stimuli obtained. The discrete-time models serve as global numerical methods on unbounded intervals for the continuous-time systems [13].

## 5.3. Impulsive Neural Networks

Most widely studied and used neural networks can be classified as either continuous or discrete. Recently, there has been a somewhat new category of neural networks which are neither purely continuous-time nor purely discrete-time. This third category of neural networks called impulsive neural networks displays a combination of characteristics of both the continuous and discrete systems [14].

To the best of our knowledge, impulsive neural networks first appeared in 1999 [14], yet I would mention that after the publication of our paper [15] in 2004 hundreds or maybe thousands of papers devoted to impulsive neural networks appear each year, mostly in China. Since it is impossible to list the most important contributions and their authors, we do not provide a list here.

## 5.4. Global Exponential Stability of Equilibrium Points of Additive Hopfield-Type Impulsive Neural Networks

### 5.4.1. Continuous-Time Case

The impulsive continuous-time neural network consists of $m$ elementary processing units (or neurons) whose state variables $x_i$ $(i = \overline{1,m})$ are governed by the system

$$\frac{dx_i(t)}{dt} = -a_i x_i(t) + \sum_{j=1}^{m} b_{ij} f_j\left(x_j(t)\right) + \sum_{j=1}^{m} c_{ij} g_j\left(x_j(t-\tau_{ij})\right) + \\ + \sum_{j=1}^{m} d_{ij} h_j\left(\int_0^{\infty} K_{ij}(s) x_j(t-s)\, ds\right) + I_i, t > 0, t \neq t_k, \tag{5.1}$$

$$\Delta x_i(t_k) = -B_{ik} x_i(t_k) + \int_{t_{k-1}}^{t_k} \psi_{ik}(s) x_i(s)\, ds + \gamma_{ik}, i = \overline{1,m}, k \in \mathbb{N}, \tag{5.2}$$

with initial values prescribed by piecewise-continuous functions $x_i(s) = \phi_i(s)$, which are bounded for $s \in (-\infty, 0]$. In (5.1), the coefficient $a_i > 0$ is the rate with which the $i$-th unit self-regulates or resets its potential when isolated from other units and inputs; $f_j(\cdot), g_j(\cdot), h_j(\cdot)$ denote activation functions; the parameters $b_{ij}, c_{ij}, d_{ij}$ are real numbers that represent the weights (or strengths) of the synaptic connections between the $j$-th unit and the $i$-th unit; the real constant $I_i$ represents an input signal introduced from outside the network to the $i$-th unit; $\tau_{ij}$ are nonnegative real numbers whose presence indicates the delayed transmission of signals at time $t-\tau_{ij}$ from the the $j$-th unit to the unit $i$; and the delay kernels $K_{ij}(s)$ incorporate the fading past effects (or fading memories) of the $j$-th unit on the $i$-th unit. In (5.2), $\Delta x_i(t_k) = x_i(t_k+0) - x_i(t_k-0)$ denote impulsive state displacements at fixed instants of time $t_k$ ($k \in \mathbb{N}$) involving integral terms whose kernels $\psi_{ik}: [t_{k-1}, t_k] \to \mathbb{R}$ are measurable functions, essentially bounded on the respective interval; $\mathbb{R}$ and $\mathbb{N}$ denote the set of all real numbers and all positive integers, respectively. Here it is assumed that the sequence of times $\{t_k\}_{k=1}^{\infty}$ satisfies $0 = t_0 < t_1 < t_2 < \cdots < t_k \to \infty$ as $k \to \infty$; the functions $x_i(t)$ are assumed continuous from the left at their points of discontinuity $t_k$; $B_{ik}$ and $\gamma_{ik}$ are some real constants.

The assumptions that accompany the impulsive network (5.1), (5.2) are given as follows:

**A1.** For the activation functions $f_j, g_j, h_j: \mathbb{R} \to \mathbb{R}$ there exist positive constants $F_j, G_j, H_j$ such that

$$F_j = \sup_{x \neq y} \left| \frac{f_j(x)-f_j(y)}{x-y} \right|, G_j = \sup_{x \neq y} \left| \frac{g_j(x)-g_j(y)}{x-y} \right|,$$

$$H_j = \sup_{x \neq y} \left| \frac{h_j(x)-h_j(y)}{x-y} \right| \text{ for } x, y \in \mathbb{R}, j = \overline{1,m}$$

**A2.** $a_i - F_i \sum_{j=1}^{m} |b_{ji}| - G_i \sum_{j=1}^{m} |c_{ji}| - H_i \sum_{j=1}^{m} |d_{ji}| > 0, i = \overline{1,m}.$

**A3.** $K_{ij}: [0,\infty) \to [0,\infty)$ $(i,j = \overline{1,m})$ are bounded and piecewise continuous.

**A4.** $\int_0^{\infty} K_{ij}(s)\, ds = 1$ $(i,j = \overline{1,m})$.

**A5.** There exists a positive number $\mu$ such that $\int_0^{\infty} K_{ij}(s) e^{\mu s}\, ds < \infty$ $(i,j = \overline{1,m})$.

An equilibrium point of the impulsive network (5.1), (5.2) is denoted by $x^* = (x_1^*, x_2^*, \dots, x_m^*)^T$, whereby the components $x_i^*$ are governed by the algebraic system

$$a_i x_i^* = \sum_{j=1}^{m} b_{ij} f_j(x_j^*) + \sum_{j=1}^{m} c_{ij} g_j(x_j^*) + \sum_{j=1}^{m} d_{ij} h_j(x_j^*) + I_i, i = \overline{1,m}, \quad (5.3)$$

and satisfy the linear equations

$$\left(-B_{ik} + \int_{t_{k-1}}^{t_k} \psi_{ik}(s)\, ds\right) x_i^* + \gamma_{ik} = 0, k \in \mathbb{N}, i = \overline{1,m} \quad (5.4)$$

**Lemma 1.** *Let conditions A1, A2 be satisfied. Then system (5.3) has a unique solution* $x^* = (x_1^*, x_2^*, \dots, x_m^*)^T$.

In other words, if conditions A1 – A4 are satisfied, the system without impulses (5.1) has a unique equilibrium point $x^* = (x_1^*, x_2^*, \dots, x_m^*)^T$.

**Proof.** In system (5.3) we perform the substitution $y_i = a_i x_i^*, i = \overline{1,m}$. Thus we obtain the system

$$y_i = \Phi_i(y) \equiv \sum_{j=1}^{m} \left[ b_{ij} f_j\left(\frac{y_j}{a_j}\right) + c_{ij} g_j\left(\frac{y_j}{a_j}\right) + d_{ij} h_j\left(\frac{y_j}{a_j}\right) \right] + I_i, i = \overline{1,m}$$

We shall show that the mapping $y \mapsto \Phi(y) = \left(\Phi_1(y), \Phi_2(y), \dots, \Phi_m(y)\right)^T$ acts as a contraction in the space $\mathbb{R}^m$ equipped with the norm $\|y\| = \sum_{i=1}^{m} |y_i|$. In fact, for any $y, z \in \mathbb{R}^m$ by virtue of A1 we have

$$|\Phi_i(y) - \Phi_i(z)| \le \sum_{j=1}^{m} \left(|b_{ij}|F_j + |c_{ij}|G_j + |d_{ij}|H_j\right) \frac{|y_j - z_j|}{a_j}, i = \overline{1,m}$$

A summation with respect to $i$ and changing the order of summation yield

$$\|\Phi(y) - \Phi(z)\| = \sum_{j=1}^{m} |\Phi_i(y) - \Phi_i(z)| \le$$

$$\le \sum_{i=1}^{m} \sum_{j=1}^{m} \left(|b_{ij}|F_j + |c_{ij}|G_j + |d_{ij}|H_j\right) \frac{|y_j - z_j|}{a_j} =$$

$$= \sum_{i=1}^{m} \frac{1}{a_i} \left(F_i \sum_{j=1}^{m} |b_{ji}| + G_i \sum_{j=1}^{m} |c_{ji}| + H_i \sum_{j=1}^{m} |d_{ji}|\right) |y_i - z_i|$$

From condition A2

$$\frac{1}{a_i} \left(F_i \sum_{j=1}^{m} |b_{ji}| + G_i \sum_{j=1}^{m} |c_{ji}| + H_i \sum_{j=1}^{m} |d_{ji}|\right) < 1, i = \overline{1,m},$$

thus

$$\alpha \equiv \max_{i=\overline{1,m}} \frac{1}{a_i} \left(F_i \sum_{j=1}^{m} |b_{ji}| + G_i \sum_{j=1}^{m} |c_{ji}| + H_i \sum_{j=1}^{m} |d_{ji}|\right) < 1,$$

and

$$\|\Phi(y) - \Phi(z)\| \le \sum_{j=1}^{m} \alpha |y_i - z_i| = \alpha \|y - z\|$$

This proves the contraction property of the mapping $\Phi$. □

Our main result in the present subsection is the following

**Theorem 1.** *Let system (5.1), (5.2) have an equilibrium point* $x^* = (x_1^*, x_2^*, \dots, x_m^*)^T$ *and satisfy the conditions A1 – A5. Then there exist constants* $M > 1$ and $\lambda > 0$ *such that all solutions* $x(t) = \left(x_1(t), x_2(t), \dots, x_m(t)\right)^T$ *of system (5.1), (5.2) satisfy the estimate*

$$\sum_{i=1}^{m} |x_i(t) - x_i^*| \le M e^{-\lambda t} \prod_{k=1}^{i(0,t)} \left\{ \max_{i=\overline{1,m}} |1 - B_{ik}| + \max_{i=\overline{1,m}} \int_{t_{k-1}}^{t_k} |\psi_{ik}(s)|\, ds \right\} \times$$

$$\times \sum_{i=1}^{m} \sup_{s \in (-\infty, 0]} |x_i(s) - x_i^*| \; for\ all\ t > 0, \qquad (5.5)$$

*where* $i(0,t) = \max\{k \in \{0\} \cup \mathbb{N} \colon t_k < t\}$ *is the number of instants of impulse effect* $t_k$ *in the interval* $(0, t)$.

**Proof.** First we notice that the equilibrium point $x^*$ is unique by virtue of condition A2 and Lemma 1.

Let us consider the functions $\Phi_i$: $[0,\mu]$ defined by

$$\Phi_i(\lambda) = a_i - \lambda - F_i \sum_{j=1}^{m} |b_{ji}| - G_i \sum_{j=1}^{m} |c_{ji}| e^{\lambda\tau_{ji}} - H_i \sum_{j=1}^{m} |d_{ji}| \int_0^{\infty} K_{ji}(s) e^{\lambda s}\, ds,$$

$i = \overline{1,m}$. We have

$$\Phi_i(0) = a_i - F_i \sum_{j=1}^{m} |b_{ji}| - G_i \sum_{j=1}^{m} |c_{ji}| - H_i \sum_{j=1}^{m} |d_{ji}| > 0,$$

by virtue of condition A2. Now, because of the assumptions A3 – A5 each $\Phi_i(\cdot)$ is well defined, continuous and decreasing on $[0,\mu]$. Thus there exists $\lambda_i^* \in (0,\mu]$ such that $\Phi_i(\lambda) > 0$ for $\lambda \in (0,\lambda_i^*), i = \overline{1,m}$. Then, choosing $\lambda^* = \min\{\lambda_1^*, \lambda_2^*, \dots, \lambda_m^*\}$, we have

$$\Phi_i(\lambda) > 0, \lambda \in (0,\lambda^*), i = \overline{1,m}. \tag{5.6}$$

We have from (5.1) and (5.3) that

$$\begin{aligned} D^+|x_i(t) - x_i^*| \le -a_i|x_i(t) - x_i^*| + \sum_{j=1}^{m} |b_{ij}|\, F_j\, |x_j(t) - x_j^*| \, + \\ + \sum_{j=1}^{m} |c_{ij}|\, G_j\, |x_j(t - \tau_{ij}) - x_j^*| \, + \\ + \sum_{j=1}^{m} |d_{ij}|\, H_j \int_0^{\infty} K_{ij}(s)\, |x_j(t-s) - x_j^*|\, ds, \end{aligned} \tag{5.7}$$

for $i = \overline{1,m}, t > 0, t \ne t_k$, where $D^+ f(t)$ denotes the upper right Dini derivative of a continuous function $f(t)$ defined by

$$D^+ f(t) = \lim_{h\to 0^+} \sup_{0<\Delta t\le h} \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

Let us note that if the continuous function $f(t)$ is differentiable at $t_0$, then

$$D^+|f(t_0)| = \begin{cases} \dot f(t_0) \text{ when } f(t_0) > 0, \\ -\dot f(t_0) \text{ when } f(t_0) < 0, \\ |\dot f(t_0)| \text{ when } f(t_0) = 0 \end{cases}$$

Next we define

$$y_i(t) = |x_i(t) - x_i^*| e^{\lambda t}, \tag{5.8}$$

where $i = \overline{1,m}, t \in \mathbb{R}$, and from (5.7) we derive

$$\begin{aligned} D^+ y_i(t) \le -(a_i - \lambda) y_i(t) + \sum_{j=1}^{m} |b_{ij}|\, F_j\, y_j(t) + \\ + \sum_{j=1}^{m} |c_{ij}|\, G_j\, e^{\lambda\tau_{ij}} y_j(t - \tau_{ij}) + \sum_{j=1}^{m} |d_{ij}|\, H_j \int_0^{\infty} K_{ij}(s)\, e^{\lambda s} y_j(t-s)\, ds, \end{aligned}$$

for $t > 0, t \ne t_k$. We define a Lyapunov functional $V(\cdot)$ by

$$V(t) = \sum_{i=1}^{m} \Big\{ y_i(t) + \sum_{j=1}^{m} |c_{ij}|\, G_j\, e^{\lambda \tau_{ij}} \int_{t-\tau_{ij}}^{t} y_j(s)\, ds +$$

$$+ \sum_{j=1}^{m} |d_{ij}|\, H_j \int_0^\infty K_{ij}(s)\, e^{\lambda s} \left( \int_{t-s}^{t} y_i(\sigma)\, d\sigma \right) ds \Big\}, t \geq 0 \qquad (5.9)$$

It is easily seen that $V(t) \geq 0$ for $t > 0$ and

$$V(0) \leq \sum_{i=1} \Big\{ y_i(0) + \sum_{j=1}^{m} |c_{ij}|\, G_j\, e^{\lambda \tau_{ij}} \sup_{s \in [-\tau, 0]} y_j(s) +$$

$$+ \sum_{j=1}^{m} |d_{ij}|\, H_j \int_0^\infty K_{ij}(s)\, e^{\lambda s} s\, ds \sup_{s \in (-\infty, 0]} y_j(s) \Big\},$$

where $\tau = \max \{ \tau_{ij} : i, j = \overline{1,m} \}$, that is

$$V(0) \leq M \sum_{i=1}^{m} \sup_{s \in (-\infty, 0]} |x_j(s) - x_i^*|, \qquad (5.10)$$

with

$$M = \max_{i=\overline{1,m}} \{ 1 + G_i \sum_{j=1}^{m} |c_{ji}| e^{\lambda \tau_{ji}} + H_i \sum_{j=1}^{m} |d_{ji}| \int_0^\infty K_{ji}(s) e^{\lambda s} s\, ds \}$$

This implies that $V(0) < \infty$ since $\int_0^\infty K_{ij}(s) e^{\lambda s} s\, ds < \infty$ for $\lambda < \mu$. Further on,

$$D^+ V(t) \leq \sum_{i=1}^{m} \{ -(a_i - \lambda) y_i(t) + \sum_{j=1}^{m} |c_{ij}|\, G_j\, e^{\lambda \tau_{ij}} y_j(t) +$$

$$+ \sum_{j=1}^{m} |d_{ij}|\, H_j \left( \int_0^\infty K_{ij}(s)\, e^{\lambda s}\, ds \right) y_j(t) \} =$$

$$= -\sum_{i=1}^{m} \{ a_i - \lambda - G_i \sum_{j=1}^{m} |c_{ji}|\, e^{\lambda \tau_{ji}} - H_i \sum_{j=1}^{m} |d_{ji}| \int_0^\infty K_{ij}(s)\, e^{\lambda s}\, ds \} y_i(t) =$$

$$= -\sum_{i=1}^{m} \Phi_i(\lambda) y_i(t) \leq 0 \text{ for } t > 0, t \neq t_k,$$

by virtue of (5.6). This implies that $V(t)$ is nonincreasing on every interval $(t_{k-1}, t_k]$, $k \in \mathbb{N}$, thus

$$V(t) \leq V(t_{k-1} + 0) \text{ for } t \in (t_{k-1}, t_k], k \in \mathbb{N} \qquad (5.11)$$

In particular,

$$V(t_k) \leq V(t_{k-1} + 0), k \in \mathbb{N} \qquad (5.12)$$

Further on, making use of the equalities (5.2) and (5.4), for an arbitrary moment of impulse effect $t_k, k \in \mathbb{N}$, we successively find

$$\Delta x_i(t_k) = -B_{ik}(x_i(t_k) - x_i^*) + \int_{t_{k-1}}^{t_k} \psi_{ik}(s)(x_i(s) - x_i^*)\, ds,$$

$$|x_i(t_k + 0) - x_i^*| \leq |1 - B_{ik}|\, |x_i(t_k) - x_i^*| + \int_{t_{k-1}}^{t_k} |\psi_{ik}(s)|\, |x_i(s) - x_i^*|\, ds,$$

$$y_i(t_k + 0) \leq |1 - B_{ik}| y_i(t_k) + \int_{t_{k-1}}^{t_k} e^{\lambda(t_k - s)} |\psi_{ik}(s)|\, y_i(s)\, ds, i = \overline{1,m}$$

Making use of (5.11) and (5.12), we obtain

$$V(t_k+0) \le$$
$$\le \max_{i=\overline{1,m}} |1-B_{ik}|\, V(t_k) + \max_{i=\overline{1,m}} \int_{t_{k-1}}^{t_k} e^{\lambda(t_k-s)} |\psi_{ik}(s)|\, ds\; V(t_{k-1}+0) \le$$
$$\le \left( \max_{i=\overline{1,m}} |1-B_{ik}| + \max_{i=\overline{1,m}} \int_{t_{k-1}}^{t_k} e^{\lambda(t_k-s)} |\psi_{ik}(s)|\, ds \right) V(t_{k-1}+0),$$

and

$$V(t) \le \prod_{k=1}^{i(0,t)} \left( \max_{i=\overline{1,m}} |1-B_{ik}| + \max_{i=\overline{1,m}} \int_{t_{k-1}}^{t_k} e^{\lambda(t_k-s)} |\psi_{ik}(s)|\, ds \right) V(0) \quad (5.13)$$

for all $t > 0$. Finally, from (5.8) and (5.9) we have

$$\sum_{i=1}^{m} |x_i(t) - x_i^*| = e^{-\lambda t} \sum_{i=1}^{m} y_i(t) \le e^{-\lambda t} V(t)$$

The last inequality combined with (5.13) and (5.10) yields (5.5). □

**Definition 1.** The equilibrium point $x^* = (x_1^*, x_2^*, \dots, x_m^*)^T$ of system (5.1), (5.2) is said to be *globally exponentially stable* (with Lyapunov exponent $\lambda$) if there exist constants $\lambda > 0$ and $M \ge 1$, and any solution $x(t) = \left(x_1(t), x_2(t), \dots, x_m(t)\right)^T$ of system (5.1), (5.2) is defined for all $t > 0$ and we have

$$\sum_{i=1}^{m} |x_i(t) - x_i^*| \le M e^{-\lambda t} \sum_{i=1}^{m} \sup_{s\in(-\infty,0]} |x_i(s) - x_i^*| \text{ for all } t \ge 0 \quad (5.14)$$

For three sets of additional assumptions on the impulse effects we will show that inequality (5.5) implies global exponential stability of the equilibrium point $x^*$ of the impulsive system (5.1), (5.2).

**Corollary 1.** *Let all conditions of Theorem 1 hold. Let there exist $\lambda \in (0, \lambda^*)$ such that*

$$\max_{i=\overline{1,m}} |1-B_{ik}| + \max_{i=\overline{1,m}} \int_{t_{k-1}}^{t_k} |\psi_{ik}(s)| e^{\lambda(t_k-s)}\, ds \le 1,$$

*for all sufficiently large values of $k \in \mathbb{N}$. Then the equilibrium point $x^*$ of the impulsive system (5.1), (5.2) is globally exponentially stable with Lyapunov exponent $\lambda$.*

The proof of this corollary is obvious. The global exponential stability is provided by the rather small magnitudes of the impulse effects. Further we will show that we may have global exponential stability for quite large and even unbounded magnitudes of the impulse effects provided that these do not occur too often.

**Corollary 2.** *Let all conditions of Theorem 1 hold and*

$$\limsup_{t\to\infty} \frac{i(0,t)}{t} = p < +\infty \quad (5.15)$$

*Let there exist positive constants $\lambda \in (0, \lambda^*)$ and $B$ satisfying the inequalities*

$$\max_{i=\overline{1,m}} |1 - B_{ik}| + \max_{i=\overline{1,m}} \int_{t_{k-1}}^{t_k} |\psi_{ik}(s)| e^{\lambda(t_k - s)}\, ds \le B, \qquad (5.16)$$

*for all sufficiently large values of $k \in \mathbb{N}$, and $p \ln B < \lambda$. Then for any $\tilde{\lambda} \in (0, \lambda - p \ln B)$ the equilibrium point $x^*$ of the impulsive system (5.1), (5.2) is globally exponentially stable with Lyapunov exponent $\tilde{\lambda}$.*

**Proof.** Inequalities (5.5) and (5.16) yield

$$\sum_{i=1}^m |x_i(t) - x_i^*| \le M e^{-\lambda t} B^{i(0,t)} \sum_{i=1}^m \sup_{s \in (-\infty, 0]} |x_i(s) - x_i^*| \text{ for all } t > 0$$

Condition (5.15) means that for any $\varepsilon > 0$ there exists $T = T(\varepsilon) > 0$ such that the inequality

$$\frac{i(0,t)}{t} \le p + \varepsilon$$

is satisfied for all $t \ge T$. For such $t$ we have $i(0,t) \le (p + \varepsilon)t$ and

$$\sum_{i=1}^m |x_i(t) - x_i^*| \le M e^{-(\lambda - (p+\varepsilon) \ln B)t} \sum_{i=1}^m \sup_{s \in (-\infty, 0]} |x_i(s) - x_i^*|$$

It suffices to choose $\varepsilon > 0$ such that $(p + \varepsilon) \ln B < \lambda$ and $\tilde{\lambda} = \lambda - (p + \varepsilon) \ln B$. Then inequality (5.14) will be satisfied with $\tilde{\lambda}$ instead of $\lambda$ and a possibly bigger constant $M$. □

**Corollary 3.** *Let all conditions of Theorem 1 hold and there exist constants $\lambda \in (0, \lambda^*)$ and $\kappa \in (0, \lambda)$ such that*

$$\max_{i=\overline{1,m}} |1 - B_{ik}| + \max_{i=\overline{1,m}} \int_{t_{k-1}}^{t_k} |\psi_{ik}(s)| e^{\lambda(t_k - s)}\, ds \le e^{\kappa(t_k - t_{k-1})}, \qquad (5.17)$$

*for all sufficiently large values of $k \in \mathbb{N}$. Then the equilibrium point $x^*$ of the impulsive system* (5.1), (5.2) *is globally exponentially stable with Lyapunov exponent $\lambda - \kappa$.*

**Proof.** By virtue of condition (5.17) for $t \in (t_k, t_{k+1}]$ inequality (5.5) implies

$$\sum_{i=1}^m |x_i(t) - x_i^*| \le M e^{-\lambda t} e^{\kappa t_k} \sum_{i=1}^m \sup_{s \in (-\infty, 0]} |x_i(s) - x_i^*|,$$

with a possibly larger constant $M$. Since $t_k < t$, we have $e^{-\lambda t} e^{\kappa t_k} < e^{-(\lambda - \kappa)t}$ and inequality (5.14) will be satisfied with $\lambda - \kappa$ instead of $\lambda$. □

A similar condition was later introduced in the paper [16].

The results of the present subsection were essentially given in our paper [15] where impulse conditions were provided for the continuous-time neural networks considered in [12]. The exposition here follows the pattern of some of our more recent papers.

### 5.4.2. Discrete-Time Case

In this subsection we formulate a discrete-time analogue of the additive Hopfield-type neural network with impulse effect (5.1), (5.2) satisfying the assumptions A1 – A5 by the semi-discretization method and investigate its global stability characteristics. Recall that the components $x_i^*$ of an equilibrium point $x^* = (x_1^*, x_2^*, \dots, x_m^*)^T$ of (5.1), (5.2) are governed by the algebraic system (5.3) and satisfy the linear equations (5.4). According to Lemma 1, if conditions A1 – A4 are satisfied, the system without impulses (5.1) has a unique equilibrium point $x^*$.

Following [12], we will obtain a discrete-time counterpart of system (5.1). Let $h > 0$ denote a uniform discretization step size and $[t/h]$ denote the greatest integer in $t/h$. For convenience, we denote $[t/h] = n, n \in \{0\} \cup \mathbb{N}$, and, by an abuse of notation, write $x_i(n)$ instead of $x_i(nh)$. Further on, we denote $\kappa_{ij} = [\tau_{ij}/h], i,j = \overline{1,m}$. Finally, we replace the integral terms $\int_0^\infty K_{ij}(s)x_j(t-s)\,ds, i,j = \overline{1,m}$ , by sums of the form $\sum_{p=1}^{\infty} \mathcal{K}_{ij}(p)x_j(n-p)$, where $p = [s/h], \mathcal{K}_{ij}(p)$ stands for $\mathcal{K}_{ij}(ph)$ and $x_j(n-p)$ for $x_j\big((n-p)h\big)$ , and the discrete kernels $\mathcal{K}_{ij}(\cdot), i,j = \overline{1,m}$, satisfy the following conditions:

**A6.** $\mathcal{K}_{ij}(p) \geq 0$ and is bounded for $p \in \mathbb{N}$.

**A7.** $\sum_{p=1}^{\infty} \mathcal{K}_{ij}(p) = 1$.

**A8.** There exists a number $\nu > 1$ such that $\sum_{p=1}^{\infty} \mathcal{K}_{ij}(p)\nu^p < \infty$.

Now, on the interval $[nh, (n+1)h]$ $(n \in \{0\} \cup \mathbb{N})$ we approximate system (5.1) by

$$\frac{dx_i(s)}{ds} = -a_i x_i(s) + \sum_{j=1}^{m} b_{ij} f_j\left(x_j(n)\right) + \sum_{j=1}^{m} c_{ij} g_j\left(x_j(n-\kappa_{ij})\right) + \\ + \sum_{j=1}^{m} d_{ij} h_j\big(\sum_{p=1}^{\infty} \mathcal{K}_{ij}(p) x_j(n-p)\big) + I_i, i = \overline{1,m} \tag{5.18}$$

We rewrite equation (5.18) in the form

$$\frac{d}{ds}\left(x_i(s)e^{a_i s}\right) = e^{a_i s}\Big(\sum_{j=1}^{m} b_{ij} f_j\left(x_j(n)\right) + \sum_{j=1}^{m} c_{ij} g_j\left(x_j(n-\kappa_{ij})\right) + \\ + \sum_{j=1}^{m} d_{ij} h_j\big(\sum_{p=1}^{\infty} \mathcal{K}_{ij}(p) x_j(n-p)\big) + I_i\Big), i = \overline{1,m},$$

and integrate it over the interval $[nh, (n+1)h]$ to obtain

$$x_i(n+1)e^{a_i(n+1)h} - x_i(n)e^{a_i nh} = \frac{e^{a_i(n+1)h} - e^{a_i nh}}{a_i}\Big(\sum_{j=1}^{m} b_{ij} f_j\left(x_j(n)\right) +$$

$$+\sum_{j=1}^{m} c_{ij} g_j\left(x_j(n-\kappa_{ij})\right)+\sum_{j=1}^{m} d_{ij} h_j\left(\sum_{p=1}^{\infty}\mathcal{K}_{ij}(p)x_j(n-p)\right)+I_i\Big),$$

or

$$x_i(n+1)=e^{-a_i h}x_i(n)+\frac{1-e^{-a_i h}}{a_i}\Big(\sum_{j=1}^{m} b_{ij} f_j\left(x_j(n)\right)+\sum_{j=1}^{m} c_{ij} g_j\left(x_j(n-\kappa_{ij})\right)+$$

$$+\sum_{j=1}^{m} d_{ij} h_j\left(\sum_{p=1}^{\infty}\mathcal{K}_{ij}(p)x_j(n-p)\right)+I_i\Big), n\in\{0\}\cup\mathbb{N}, i=\overline{1,m} \quad (5.19)$$

This system is the discrete-time analogue of the system without impulses (5.1). It is provided with initial values of the form $x_i(-\ell)=\varphi_i(-\ell)$ $(\ell\in\{0\}\cup\mathbb{N})$, where the sequences $\{\varphi_i(-\ell)\}_{\ell=0}^{\infty}$ are bounded for all $i=\overline{1,m}$. The method used here is called *semi-discretization* [12].

An equilibrium $(x_1^*,x_2^*,\dots,x_m^*)^T$ of (5.19) satisfies the system

$$\frac{1-e^{-a_i h}}{a_i}\left\{a_i x_i^*-\left(\sum_{j=1}^{m} b_{ij} f_j(x_j^*)+\sum_{j=1}^{m} c_{ij} g_j(x_j^*)+\sum_{j=1}^{m} d_{ij} h_j(x_j^*)+I_i\right)\right\}=0,$$

$$i=\overline{1,m} \quad (5.20)$$

Obviously, the quantities

$$\phi_i(h)=\frac{1-e^{-a_i h}}{a_i}, i=\overline{1,m},$$

satisfy $\phi_i(h)>0$, thus (5.20) implies (5.3), i.e., the equilibria of the systems (5.1) and (5.19) coincide. We write the system (5.19) in the form

$$x_i(n+1)=e^{-a_i h}x_i(n)+\phi_i(h)\Big(\sum_{j=1}^{m} b_{ij} f_j\left(x_j(n)\right)+\sum_{j=1}^{m} c_{ij} g_j\left(x_j(n-\kappa_{ij})\right)+$$

$$+\sum_{j=1}^{m} d_{ij} h_j\left(\sum_{p=1}^{\infty}\mathcal{K}_{ij}(p)x_j(n-p)\right)+I_i\Big), n\in\{0\}\cup\mathbb{N}, i=\overline{1,m} \quad (5.21)$$

In [12] no restrictions were imposed on the step size $h$. Neither are such restrictions required to obtain the stability result for system (5.21).

However, we are investigating the impulsive system (5.1), (5.2). We find it appropriate to assume there is not more than one instant of impulse effect in a step. To this end we suppose that

$$\theta=\inf_{k\in\mathbb{N}}(t_{k+1}-t_k)>0, \quad (5.22)$$

and $h>0$ satisfies

$$h<\theta \quad (5.23)$$

We denote $[t_k/h]=n_k, k\in\mathbb{N}$, and obtain a sequence of positive integers $\{n_k\}_{k=1}^{\infty}$ satisfying $n_1<n_2<\dots<n_k\to\infty$. With each such integer $n_k$ we associate two values of the solution $x(n)$, namely, $x(n_k)$ which can be regarded as the value of the solution

before the impulse effect and whose components are evaluated by equations (5.21), and $x(n_k^+)$ which can be regarded as the value of the solution after the impulse effect and whose components are evaluated by the equations

$$x_i(n_k^+) - x_i(n_k) = \sum_{\ell=n_{k-1}+1}^{n_k} B_{ik\ell} x_i(\ell) + \gamma_{ik}, i = \overline{1,m}, k \in \mathbb{N}, \qquad (5.24)$$

where, for convenience, $n_0 = -1$ and $B_{ik\ell}$ are suitably chosen constants.

Further on we will call system (5.21), (5.24) the discrete-time analogue of the system with impulses (5.1), (5.2). Whenever a value $x_i(n_k)$ appears in the right-hand side of (5.21), we mean $x_i(n_k^+)$.

The components of an equilibrium point $x^* = (x_1^*, x_2^*, \dots, x_m^*)^T$ of system (5.21), (5.24) must satisfy the equations (5.3) and

$$\sum_{\ell=n_{k-1}+1}^{n_k} B_{ik\ell} x_i(\ell) + \gamma_{ik} = 0 \qquad (5.25)$$

To ensure that systems (5.1), (5.2) and (5.21), (5.24) have the same equilibrium points if any, we choose the constants $B_{ik\ell}$ so that

$$\sum_{\ell=n_{k-1}+1}^{n_k} B_{ik\ell} = -B_{ik} + \int_{t_{k-1}}^{t_k} \psi_{ik}(s)\, ds, i = \overline{1,m}, k \in \mathbb{N}$$

Our main result in the present subsection is the following:

**Theorem 2.** *Let system (5.21), (5.24) satisfy conditions A1, A2, A6 – A8 and the components of the unique equilibrium point $x^* = (x_1^*, x_2^*, \dots, x_m^*)^T$ of system (5.21) satisfy (5.25). Then there exist constants $M > 1$ and $\lambda \in (1, \nu)$ and any other solution $x(n) = (x_1(n), x_2(n), \dots, x_m(n))^T$ of system (5.21), (5.24) is defined for all $n \in \mathbb{N}$ and satisfies the estimate*

$$\sum_{i=1}^{m} \frac{|x_i(n) - x_i^*|}{\phi_i(h)} \le M\lambda^{-n} \prod_{k=1}^{i(1,n)} B_k(\lambda) \sum_{i=1}^{m} \sup_{\ell \in \{0\} \cup \mathbb{N}} \frac{|x_i(-\ell) - x_i^*|}{\phi_i(h)}, \qquad (5.26)$$

*where* $i(1,n) = \begin{cases} 0, & n \le n_1, \\ \max\{k \in \mathbb{N} : n_k < n\}, & n > n_1, \end{cases}$ *and* $B_k(\lambda) = \max_{i=\overline{1,m}} |1 + B_{ikn_k}| + \sum_{\ell=n_{k-1}+1}^{n_k - 1} \max_{i=\overline{1,m}} |B_{ik\ell}| \lambda^{n_k - \ell}, k \in \mathbb{N}$.

**Proof.** From the conditions of the theorem it follows that system (5.21), (5.24) has a unique equilibrium point $x^* = (x_1^*, x_2^*, \dots, x_m^*)^T$.

Let us consider the functions $\Phi_i : [1, \nu]$ defined by

$$\Phi_i(\lambda) = 1 - \lambda e^{-a_i h} -$$
$$-\phi_i(h)\left[F_i \sum_{j=1}^{m} |b_{ji}| + G_i \sum_{j=1}^{m} |c_{ji}| \lambda^{\kappa_{ji}+1} + H_i \sum_{j=1}^{m} |d_{ji}| \sum_{p=1}^{\infty} \mathcal{K}_{ji}(p) \lambda^{p+1}\right], i = \overline{1,m}.$$

By virtue of conditions A7 and A2 we have

$$\Phi_i(1) = 1 - e^{-a_i h} - \phi_i(h)\left[F_i \sum_{j=1}^m |b_{ji}| + G_i \sum_{j=1}^m |c_{ji}| + H_i \sum_{j=1}^m |d_{ji}|\right] =$$
$$= \phi_i(h)\left[a_i - F_i \sum_{j=1}^m |b_{ji}| - G_i \sum_{j=1}^m |c_{ji}| - H_i \sum_{j=1}^m |d_{ji}|\right] > 0.$$

Now, because of the assumptions A6 and A8 each $\Phi_i(\cdot)$ is well defined, continuous and decreasing on $[1,\nu]$. Thus, there exists $\lambda_i^* \in (1,\nu]$ such that $\Phi_i(\lambda) > 0$ for $\lambda \in [1,\lambda_i^*), i = \overline{1,m}$. Choosing $\lambda^* = \min\{\lambda_1^*, \lambda_2^*, \dots, \lambda_m^*\}$, we have

$$\Phi_i(\lambda) > 0, \lambda \in [1,\lambda^*), i = \overline{1,m}. \tag{5.27}$$

From (5.21) and (5.20) it follows that

$$|x_i(n+1) - x_i^*| \le e^{-a_i h}|x_i(n) - x_i^*| + \phi_i(h) \sum_{j=1}^m \Big[ |b_{ij}|\, F_j\, |x_j(n) - x_j^*| +$$
$$+ |c_{ij}|\, G_j\, |x_j(n - \kappa_{ij}) - x_j^*| + |d_{ij}|\, H_j \sum_{p=1}^\infty \mathcal{K}_{ij}(p) |x_j(n-p) - x_j^*|\Big], \tag{5.28}$$

$i = \overline{1,m}$. If for any $\lambda \in (0,\lambda^*)$ we denote

$$y_i(n) = \lambda^n \frac{|x_i(n) - x_i^*|}{\phi_i(h)}, i = \overline{1,m}, n \in \mathbb{Z}, \tag{5.29}$$

where $\mathbb{Z}$ is the set of all integers, then from (5.28) we have

$$y_i(n+1) \le \lambda e^{-a_i h} y_i(n) + \phi_i(h) \sum_{j=1}^m \Big[|b_{ij}|\, F_j\, y_j(n) +$$
$$+ |c_{ij}|\, G_j \lambda^{\kappa_{ij}+1} y_j(n - \kappa_{ij}) + |d_{ij}|\, H_j \sum_{p=1}^\infty \mathcal{K}_{ij}(p) \lambda^{p+1} y_j(n-p)\Big], \tag{5.30}$$

for $n \in \{0\} \cup \mathbb{N}$ and $i = \overline{1,m}$ . Next consider a Lyapunov functional $V(n) = V(y_1, y_2, \dots, y_m)(n)$ defined by

$$V(n) = \sum_{i=1}^m \Big\{ y_i(n) + \sum_{j=1}^m \phi_j(h) \Big[ |c_{ij}|\, G_j \lambda^{\kappa_{ij}+1} \sum_{\ell=n-\kappa_{ij}}^{n-1} y_j(\ell) +$$
$$+ |d_{ij}|\, H_j \sum_{p=1}^\infty \mathcal{K}_{ij}(p) \lambda^{p+1} \sum_{r=n-p}^{n-1} y_j(r) \Big] \Big\}, n \in \{0\} \cup \mathbb{N} \tag{5.31}$$

It is easy to see that $V(n) \ge 0$ and $V(0) < \infty$ by A8. More precisely,

$$V(0) \le M \sum_{i=1}^m \sup_{\ell \in \{0\} \cup \mathbb{N}} \frac{|x_i(-\ell) - x_i^*|}{\phi_i(h)}, \tag{5.32}$$

where

$$M = \max_{i=\overline{1,m}} \Big\{ 1 + \phi_i(h) \Big[ G_i \sum_{j=1}^m |c_{ji}| \sum_{\ell=1}^{\kappa_{ji}} \lambda^\ell + H_i \sum_{j=1}^m |d_{ji}| \sum_{p=1}^\infty \mathcal{K}_{ji}(p) \sum_{r=1}^p \lambda^r \Big] \Big\}$$

Further on, by virtue of (5.30) we obtain

$$V(n+1) \le \sum_{i=1}^m \Big\{ \lambda e^{-a_i h} y_i(n) + \sum_{j=1}^m \phi_j(h) \Big[ |b_{ij}|\, F_j\, y_j(n) +$$
$$+ |c_{ij}|\, G_j \lambda^{\kappa_{ij}+1} y_j(n-\kappa_{ij}) + |d_{ij}|\, H_j \sum_{p=1}^\infty \mathcal{K}_{ij}(p) \lambda^{p+1} y_j(n-p) \Big] +$$
$$+ \sum_{j=1}^m \phi_j(h) \left[ \begin{array}{c} |c_{ij}|\, G_j \lambda^{\kappa_{ij}+1} \sum_{\ell=n+1-\kappa_{ij}}^{n} y_j(\ell) + \\ + |d_{ij}|\, H_j \sum_{p=1}^\infty \mathcal{K}_{ij}(p) \lambda^{p+1} \sum_{r=n+1-p}^{n} y_j(r) \end{array} \right] \Bigg\} =$$

$$= \sum_{i=1}^{m}\{\lambda e^{-a_i h} y_i(n) + \sum_{j=1}^{m} \phi_j(h)[|b_{ij}|\, F_j\, y_j(n) +$$

$$+|c_{ij}|\, G_j \lambda^{\kappa_{ij}+1} \sum_{\ell=n-\kappa_{ij}}^{n} y_j(\ell) + |d_{ij}|\, H_j \sum_{p=1}^{\infty} \mathcal{K}_{ij}(p)\lambda^{p+1} \sum_{r=n-p}^{n} y_j(r)]\},$$

and

$$V(n+1) - V(n) \le \sum_{i=1}^{m}\{(\lambda e^{-a_i h} - 1) y_i(n) + \sum_{j=1}^{m} \phi_j(h)[|b_{ij}|\, F_j\, y_j(n) +$$

$$+|c_{ij}|\, G_j \lambda^{\kappa_{ij}+1} y_j(n) + |d_{ij}|\, H_j \sum_{p=1}^{\infty} \mathcal{K}_{ij}(p)\lambda^{p+1} y_j(n)]\} =$$

$$= -\sum_{j=1}^{m}\{1 - \lambda e^{-a_i h} - \phi_i(h)[F_i \sum_{j=1}^{m}|b_{ji}| + G_i \sum_{j=1}^{m}|c_{ji}|\lambda^{\kappa_{ij}+1} +$$

$$+H_j \sum_{j=1}^{m}|d_{ji}| \sum_{p=1}^{\infty} \mathcal{K}_{ji}(p)\lambda^{p+1}]\}\, y_i(n) = -\sum_{i=1}^{m} \Phi_i(\lambda)\, y_i(n) \le 0,$$

by virtue of (5.27). This implies $V(n+1) \le V(n)$ for $n \ne n_k$ and $(n_k+1) \le V(n_k^+)$, where $V(n_k^+)$ contains $|x_i(n_k^+) - x_i^*|$ instead of $|x_i(n_k) - x_i^*|$. The above inequalities yield

$$V(n) \le \begin{cases} V(n_k^+) & \text{for} \quad n_k < n \le n_{k+1}, \\ V(0) & \text{for} \quad 0 < n \le n_1 \end{cases} \tag{5.33}$$

Further on, making use of equalities (5.24) and (5.25), for any $k \in \mathbb{N}$ we find successively

$$|x_i(n_k^+) - x_i^*| \le |1 + B_{ikn_k}|\, |x_i(n_k) - x_i^*| + \sum_{\ell=n_{k-1}+1}^{n_k-1} |B_{ik\ell}|\, |x_i(\ell) - x_i^*|,$$

$$y_i(n_k^+) \le |1 + B_{ikn_k}| y_i(n_k) + \sum_{\ell=n_{k-1}+1}^{n_k-1} |B_{ik\ell}|\, \lambda^{n_k-\ell}\, y_i(\ell),$$

$$V(n_k^+) \le \max_{i=\overline{1,m}} |1 + B_{ikn_k}| V(n_k) + \sum_{\ell=n_{k-1}+1}^{n_k-1} \max_{i=\overline{1,m}} |B_{ik\ell}|\, \lambda^{n_k-\ell}\, V(\ell),$$

thus $V(n_k^+) \le B_k(\lambda) V(n_{k-1}^+)$ for $k \ge 2$ and $V(n_1^+) \le B_1(\lambda) V(0)$, where the quantities $B_k(\lambda)$ were introduced in the statement of Theorem 2.

Combining the last inequalities and (5.23), we derive the estimate

$$V(n) \le \prod_{k=1}^{i(1,n)} B_k(\lambda)\, V(0) \tag{5.34}$$

Finally, from the inequalities

$$\sum_{i=1}^{m} \frac{|x_i(n) - x_i^*|}{\phi_i(h)} \le \lambda^{-n} V(0),$$

(5.34) and (5.32) we deduce (5.26) for any $\lambda \in (1, \lambda^*)$. □

**Definition 2.** The equilibrium point $x^* = (x_1^*, x_2^*, \dots, x_m^*)^T$ of system (5.21), (5.24) is said to be *globally exponentially stable with multiplier* $\rho$ if there exist constants $M \ge 1$ and $\rho \in (0,1)$ and any other solution $x(n) = (x_1(n), x_2(n), \dots, x_m(n))^T$ of system (5.21), (5.24) is defined for all $n \in \mathbb{N}$ and satisfies the estimate

$$\sum_{i=1}^{m} \frac{|x_i(n)-x_i^*|}{\phi_i(h)} \leq M\rho^n \sum_{i=1}^{m} \sup_{\ell \in \{0\}\cup\mathbb{N}} \frac{|x_i(-\ell)-x_i^*|}{\phi_i(h)} \text{ for all } n \in \{0\}\cup\mathbb{N} \tag{5.35}$$

For three sets of additional assumptions on the impulse effects we will show that inequality (5.26) implies global exponential stability of the equilibrium point $\boldsymbol{x}^*$ of the system (5.21), (5.24).

**Corollary 4.** *Let all conditions of Theorem 2 hold. Let there exist* $\lambda = (1, \lambda^*)$ *such that* $B_k(\lambda) \leq 1$ *for all sufficiently large values of* $k \in \mathbb{N}$. *Then the equilibrium point* $\boldsymbol{x}^*$ *of the discrete-time system (5.21), (5.24) is globally exponentially stable with multiplier* $1/\lambda$.

The proof of this corollary is obvious. The global exponential stability is provided by the rather small magnitudes of the impulse effects. Further we will show that we may have global exponential stability for quite large and even unbounded magnitudes of the impulse effects provided that these do not occur too often.

**Corollary 5.** *Let all conditions of Theorem 2 hold and*

$$\limsup_{n\to\infty} \frac{i(1,n)}{n} = p < \infty \tag{5.36}$$

*Let there exist positive constants* $\lambda = (1, \lambda^*)$ *and* $B$ *satisfying the inequalities*

$$B_k(\lambda) \leq B, \tag{5.37}$$

*for all sufficiently large values of* $k \in \mathbb{N}$, *and* $B^p < \lambda$. *Then for any* $\rho \in \left(\frac{B^p}{\lambda}, 1\right)$ *the equilibrium point* $\boldsymbol{x}^*$ *of the discrete-time system (5.21), (5.24) is globally exponentially stable with multiplier* $\rho$.

**Proof.** Inequalities (5.26) and (5.37) yield

$$\sum_{i=1}^{m} \frac{|x_i(n)-x_i^*|}{\phi_i(h)} \leq M\lambda^{-n} B^{i(1,n)} \sum_{i=1}^{m} \sup_{\ell \in \{0\}\cup\mathbb{N}} \frac{|x_i(-\ell)-x_i^*|}{\phi_i(h)} \text{ for all } n \in \mathbb{N}$$

Condition (5.36) means that for any $\varepsilon > 0$ there exists $N = N(\varepsilon) > 0$ such that the inequality

$$\frac{i(1,n)}{n} \leq p + \varepsilon$$

is satisfied for all $n \geq N$. For such $n$ we have $i(1,n) \leq (p+\varepsilon)n$ and

$$\sum_{i=1}^{m} \frac{|x_i(n)-x_i^*|}{\phi_i(h)} \leq M\left(\frac{B^{p+\varepsilon}}{\lambda}\right)^n \sum_{i=1}^{m} \sup_{\ell \in \{0\}\cup\mathbb{N}} \frac{|x_i(-\ell)-x_i^*|}{\phi_i(h)}$$

It suffices to choose $\varepsilon > 0$ such that $B^{p+\varepsilon} < \lambda$ and $\rho = \frac{B^{p+\varepsilon}}{\lambda}$. Then inequality (5.35) will be satisfied with a possibly bigger constant $M$. □

**Corollary 6.** *Let all conditions of Theorem 1 hold and there exist constants $\lambda = (1, \lambda^*)$ and $\mu \in (1, \lambda)$ such that*

$$B_k(\lambda) \leq \mu^{n_k - n_{k-1}}, \tag{5.38}$$

*for all sufficiently large values of $k \in \mathbb{N}$. Then the equilibrium point $x^*$ of the discrete-time system (5.21), (5.24) is globally exponentially stable with multiplier $\mu/\lambda$.*

**Proof.** By virtue of condition (5.38) for $n_k < n \leq n_{k+1}$ inequality (5.26) implies

$$\sum_{i=1}^{m} \frac{|x_i(n) - x_i^*|}{\phi_i(h)} \leq M\lambda^{-n}\mu^{n_k} \sum_{i=1}^{m} \sup_{\ell \in \{0\} \cup \mathbb{N}} \frac{|x_i(-\ell) - x_i^*|}{\phi_i(h)},$$

with a possibly larger constant $M$. Since $n_k < n$, we have $\lambda^{-n}\mu^{n_k} < \left(\frac{\mu}{\lambda}\right)^n$ and inequality (5.35) will be satisfied with $\rho = \mu/\lambda$. □

These results were first reported in a very concise form at the Conference on Differential Equations and Applications, Žilina, Slovakia, 2003, and published in [17]. The results of the present subsection were essentially given in our paper [18] where impulse conditions were provided for the continuous-time neural networks considered in [12]. The exposition here follows the pattern of some of our more recent papers. In particular, the discretization of the impulse conditions used here is different from the one used in [18].

## 5.5. Conclusion

We first provided a short survey on artificial neural networks and listed some of their applications. Next we concentrated on Hopfield-type neural networks with impulses. For impulsive continuous-time neural networks of Hopfield type with both constant and infinite distributed delays sufficient conditions were found for the existence of a unique equilibrium point and its global exponential stability. Discrete-time counterparts of the aforementioned neural networks were formulated by the semi-discretization method, and sufficient conditions were found for the global exponential stability of the unique equilibrium point.

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